Stability Analysis of Stage–Structured Prey with Selective Harvesting in a Predator–Prey Model

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Abstract

This research examines the dynamics of a two-species model in the context of harvesting. One species is divided into immature and mature prey subpopulations, while the other is categorised as a predator. The various types of functional responses between these two species have been incorporated. Holling type–I pertains to immature prey due to its ease of capture, whereas Holling type–II is applicable to mature prey. Additionally, mature prey has been harvested. The concepts of boundedness and positivity are investigated. It has been determined that every potential equilibrium point exists. Using the Jacobian matrix, stability analysis is performed. Numerical simulations are done using MATLAB software.

Keywords: equilibrium points; functional response; Jacobian matrix; population dynamics; stability analysis; stage structure model.

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Malaysian J. Math. Sci. 19(4): 1507–1524(2025) *1507 - 1524*

1 Introduction

In recent years, many researchers have become more interested in theoretical ecology due to its rich dynamics. In particular, the interaction between the species and the resources plays a vital role. Firstly, Lotka and Volterra developed the interaction between two species of prey and predator. Many mathematical models between prey and predator have been developed recently. These models were formulated depending on the size, growth, age, stage, death, environmental resources, harvesting, immigration and emigration. Stage-structured models are developed depending on the various stages. Stage refers to a period of individual life history (juvenile, adult / immature, mature). The organism may continue to develop at a stage or go on to others [4]. Stage–structured predator–prey systems incorporate the stages of species growth. Generally, the species have stage structure in their population and there is significant interaction between the species. A multitude of authors have examined the stage-structured paradigm [1, 2]. The influence of cannibalism among predators in a predator-prey system leads to destabilisation [18]. Using the data, the several elements influencing the stage-structured model have been studied [28] . A function response is the intake rate of predator for prey density [10]. Numerous functional responses are utilized for various models, including the debeddington functional response, Holling's type I, II, III, and IV functional responses and the crowley martin functional response.

The influence of refuge combined with additional food within the context of a Holling type–II functional response has been established [9]. There is a literal discussion of the stage–structured prey-predator paradigm with various functional responses. Using the iterative approach with delay, Chen et al. [5] investigated the global stability of a stage–structured prey–predator system. The stage–structured model for predators was created by Pratama et al. [25] utilizing the Holling type–I functional response to examine the model's dynamics. Devi and Saikia [6] used ratio–dependent functional response and Holling type–II functional response to investigate the dynamical behavior of a stage–structured model for prey. An adult prey's anti–predator behavior was combined with a ratio–dependent functional response to investigate the global stability of a stage–structured model for prey by Panja et al. [24].

Bahlool [3] examined the dynamics of the stage–structured prey–predator model incorporating anti–predator behavior and the occurrence of bifurcations. Pandey et al. [21] investigated the emergence of various bifurcations in a stage–structured prey–predator model featuring cooperative behavior in both species, incorporating the element of delay. Zhang et al. [32] formulated the stage structured model for prey along with the harvesting. Dubey et al. [7] developed the stage structured model for predator with optimal harvesting using crowley martin functional response. Gakkhar and Gupta [8] studied the existence of hopf bifurcation in a stage structured prey using modified Holling type–II functional response. A delay stage–structured prey-predator model with hopf bifurcation and a predator–dependent functional response was proposed [30]. Kar and Chattopadhyay [12] developed the stage–structured model for predator with harvesting and time delay.

The presence of hopf bifurcation involving time delay in the stage–structured prey model utilizing Holling type functional response was examined [22]. The impact of harvesting and the behavior of stage structured prey has been studied [17]. Yadav et al. [29] conducted a study to determine the presence of hopf bifurcation in a prey model with stage structure. Holling type–IV and II functional responses are used to carry out the interaction between the species in the stage–structured model that was developed for both the predator and prey [20]. Meng et al. [19] studied the permanence of the stage–structured model for prey using crowley martin functional response. The stage structured model for predator with time delay using debeddington functional response has been formulated [16]. Li et al. [15] studied the bifurcation of stage–structured for

prey with time delay. Lan et al. [14] developed a fractional order delay stage structure model for prey, incorporating the influences of dread and refuge. The effect of refuge in prey and partially dependent predator stage structure model was investigated [11]. The influence of the Allee effect on predator populations, using a Holling type functional response between two prey species, has been examined [13].

The stage–structured model for single species along with harvesting were developed by Song and Chen [27]. Yu et al. [31] developed the single species stage structured model with Michalelis–Menten type harvesting. Panigoro et al. [23] formulated the prey predator model with complexity in age structure, memory effect and harvesting. The dynamical behavior of stage structured model for prey using various functional response is numerically explained [26]. Numerous research works have integrated the dynamic relationships of a prey–predator model with a stage structure, featuring a range of functional responses, refuges, Allee effects, interspecific competition, and delays of different kinds. The previously utilized models were predicated on stage structure, focusing either on prey or predator. The model is now established based on the stage structure of prey, incorporating two functional responses: Holling type–II and Holling type–I. Holling type–I is utilized for interactions between immature prey and predators, whilst Holling type–II is employed for interactions between mature prey and predators. The Holling type–I functional response is characterised by a linear correlation between the rate of predator consumption and prey density. It follows that predators will keep eating food until they reach a saturation point as prey density increases. It is a particularly useful paradigm for systems with minimal or no handling time. The Holling type–II functional response involves the predator spending more time handling the prey at high densities and more time seeking for the prey at low densities. The decelerating intake rate that characterises the type–II functional response is a result of the idea that the consumer's ability to process food is limited. In this case, food processing and food seeking are mutually exclusive activities.

The main objective of the paper is to study the dynamical behavior of a stage–structured model for prey with harvesting. The prey population is categorized into two subgroups: immature prey and mature prey. The harvesting is done only in the mature prey. An analysis is done on the system's local stability. The existence of equilibrium points is necessary for the system to remain stable. The Jacobian matrix is utilized to analyze the model's behavior. The Routh Hurwitz Criteria are used to assess the overall stability of the system. The key contribution and novelty of this paper is given as follows:

1. The model categorises the prey population into immature and adult stages, hence enhancing biological realism compared to single-population models.

2. Modelling the predator-prey relationship at both the immature and mature stages reveals intricate ecological relationships.

3. Using Jacobian matrices and Lyapunov functions, the stability of equilibrium points is examined in order to determine when populations continue to exist or become extinct.

4. Increasing the rate of selective harvesting can cause system instability, which can result in oscillatory dynamics or the extinction of prey or predators, according to the results.

5. This work is interesting because it examines the dynamics of stage-strucured prey, where the idea of harvesting has been crucial.

This is the arrangement of the paper: Section 2 explains the mathematical model. Section 3 addresses positivity and boundedness. Section 4 examines the presence of equilibrium points. Section 5 addresses stability analysis. Section 7 delineates the numerical simulation. Ultimately, it concludes with a conclusion.

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2   Model of the System

Let us examine the following assumptions before presenting the model. We examine two species: prey and predator. Here, the prey density is divided into two categories: immature and mature.

1. Let us represent the density of the predator as $z(t)$, the density of the mature prey as $y(t)$, and the density of the immature prey as $x(t)$.

2. Here, the predator consumes both immature and mature prey.

3. With a carrying capacity of k and an intrinsic growth rate of r, the immature prey logistically grows in the absence of a predator.

4. For predation between immature prey and predator Holling type–I is used while between mature prey and predator Holling type–II is used.

5. Further, the harvesting has been done in the mature prey.

The assumptions made on the above is considered and the model of the system is framed as;

$$
\begin{aligned}
\frac{dx}{dt} &= xr\left(1 - \frac{x}{k}\right) - \beta x - \alpha xz = x\, f_1\,(x, y, z)\,, \\
\frac{dy}{dt} &= \beta x - \frac{\varepsilon yz}{1 + my} - \mu_1 y - hy = y\, f_2\,(x, y, z)\,, \\
\frac{dz}{dt} &= \alpha_1 xz + \frac{\varepsilon_1 yz}{1 + my} - \mu_2 z = z\, f_3\,(x, y, z)\,,
\end{aligned}
\tag{1}
$$

where $x\,(0) = x_0,\, y\,(0) = y_0,\, z\,(0) = z_0$. From the above system of (1), we see that the interaction functions $f_i(i = 1, 2, 3)$ are continuous and have continuous partial derivatives on the state space $R_+^3 = \{(x, y, z) \in R^3 : x \geq 0, y \geq 0, z \geq 0\}$. Indeed, they are Lipschitzian on R_+^3 and thus the fundamental theorem of existence and uniqueness assures existence and uniqueness of solution of the model of system (1) with the given initial condition.

The parameters $r, \beta, \alpha, \alpha_1, h, \varepsilon, \varepsilon_1, m, k, \mu_1$ and μ_2 are positive. The following Table 1 is a description of the parameters,

Table 1: Description of the parameters.

Parameters	Description
r	intrinsic growth rate, $r > 0$
k	carrying capacity
β	rate of maturity
α	predation rate of immature prey
α_1	rate of changeover from immature prey to predator
h	harvesting
m	handling time
ε	predation rate of mature prey
ε_1	rate of transformation from mature prey to predator
μ_1	the mortality rate of mature prey
μ_2	the mortality rate of predator

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3 Positivity and Boundedness

Theorem 3.1. *All solutions of system* (1) *in R_+^3 remain positive forever.*

Proof. System (1) positivity can be verified by the following equations,

$$x(t) = x(0) \exp \left[\int_0^t \left\{ r \left(1 - \frac{x(s)}{k} \right) - \beta - \alpha z(s) \right\} ds \right] \Rightarrow x(t) > 0,$$

$$y(t) = y(0) \exp \left[\int_0^t \left\{ \frac{\beta x(s)}{y(s)} - \frac{\varepsilon z(s)}{1 + my(s)} - \mu_1 - h \right\} ds \right] \Rightarrow y(t) > 0,$$

$$z(t) = z(0) \exp \left[\int_0^t \left\{ \alpha_1 x(s) + \frac{\varepsilon_1 y(s)}{1 + my(s)} - \mu_2 \right\} ds \right] \Rightarrow z(t) > 0.$$

Thus, the solution of the system are all positive and lies in the region R_+^3. $\qquad\square$

Theorem 3.2. *Every solution found in system* (1) *has a uniform boundary.*

Proof. Assume that any solutions for system (1) are $x(t)$, $y(t)$ and $z(t)$. Considering that,

$$\frac{dx}{dt} \leq rx \left(1 - \frac{x}{k} \right),$$

we have $\underset{t \to \infty}{Lt} \sup x(t) \leq k$.

Consider the function $U(t) = lx(t) + y(t) + z(t)$ to demonstrate the boundedness.

Taking the derivatives of system (1), we have

$$\frac{dU}{dt} = lrx \left(1 - \frac{x}{k} \right) - \mu_1 y - hy - \mu_2 z.$$

Now, choose a constant $\gamma > 0$, such that,

$$\frac{dU}{dt} + \gamma U \leq x \left(lr \left(1 - \frac{x}{k} \right) + \gamma l \right) - (\mu_1 - \gamma)y - (\mu_2 - \gamma)z.$$

Let us choose $\gamma = \min \{\mu_1, \mu_2\}$, then,

$$\frac{dU}{dt} + \gamma U \leq x \left(lr \left(1 - \frac{x}{k} \right) + \gamma l \right).$$

The maximum of $xlr \left(1 - \frac{x}{k} \right) + x\gamma l$ is $lk \left(\frac{r}{4} + \frac{\gamma}{2} \right)$. Therefore, $\frac{dU}{dt} + \gamma U \leq lk \left(\frac{r}{4} + \frac{\gamma}{2} \right) = M > 0.$

By using Gronwalls inequality, we have

$$0 \leq U(t) \leq \frac{M}{\gamma} \left(1 - e^{-\gamma t} \right) + U \left(x\left(0\right), y\left(0\right), z\left(0\right) \right).$$

Thus, when the limit $t \to \infty$ implies $0 \leq U(t) \leq \frac{M}{\gamma}$.

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Consequently, all solutions of system (1) reside within the region,

$$R = \left\{ (x, y, z) \in R_+^3 : 0 \le U(t) \le \frac{M}{\gamma} \right\}.$$

$\square$

4 Existence of Equilibrium Points

The possible existence of equilibrium points of system (1) are determined and are as follows:

1. Trivially, the equilibrium point $E_0\,(0, 0, 0)$ always exists.

2. When there is no mature prey and predator, the equilibrium point $E_1\left(k\left(1 - \frac{\beta}{r}\right), 0, 0\right)$ is present axially. This equilibrium point always exists only if $k, r, \beta > 0$ are positive.

3. When immature prey is not present, the equilibrium point $E_2(0, y^*, z^*)$ exists, where

$$y^* = \frac{\mu_2}{\varepsilon_1 - m\mu_2}, \quad \text{and} \quad z^* = -\frac{\varepsilon_1\,(\mu_1 + h)}{\varepsilon(\varepsilon_1 - m\mu_2)}.$$

This equilibrium point exists only if it satisfies the condition that $\varepsilon_1 > m\mu_2$.

4. When the predator is not present, there is an equilibrium point $E_3\,(x^*, y^*, 0)$ exists where,

$$x^* = k\left(1 - \frac{\beta}{r}\right), \quad \text{and} \quad y^* = \frac{\beta k\,(r - \beta)}{r(\mu_1 + h)}.$$

This equilibrium point exists only if it satisfies the condition that $r > \beta$.

5. The point of internal equilibrium $E_4\,(x^*, y^*, z^*)$ exists where,

$$x^* = k\left(1 - \left(\frac{\alpha z^* + \beta}{r}\right)\right),$$

$$y^* = \frac{\mu_2 - \alpha_1 x^*}{\varepsilon_1 - m\,(\mu_2 - \alpha_1 x^*)},$$

$$z^* = \frac{(1 + my^*)}{\varepsilon y} \left[\beta x^* - \mu_1 y^* - hy^*\right],$$

are found. This equilibrium point exists only if it satisfies the condition that $\mu_2 > \alpha_1 x^*$.

5 Stability Analysis

The stability of system (1) can be investigated using Jacobian matrix. The Jacobian matrix is of the form,

$$J\,(x, y, z) = \begin{bmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \\[2mm] \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} & \dfrac{\partial g}{\partial z} \\[2mm] \dfrac{\partial h}{\partial x} & \dfrac{\partial h}{\partial y} & \dfrac{\partial h}{\partial z} \end{bmatrix},$$

where

$$f(x, y, z) = rx \left(1 - \frac{x}{k}\right) - \beta x - \alpha x z,$$

$$g(x, y, z) = \beta x - \frac{\varepsilon y z}{1 + m y} - \mu_1 y - h y,$$

$$h(x, y, z) = \alpha_1 x z + \frac{\varepsilon_1 y z}{1 + m y} - \mu_2 z.$$

System (1) is linearized using Jacobian Matrix and is given below,

$$J(x, y, z) = \begin{bmatrix} r\left(1 - \dfrac{2x}{k}\right) - \beta - \alpha z & 0 & -\alpha x \\[2mm] \beta & \dfrac{-\varepsilon z}{(1 + m y)^2} - \mu_1 - h & \dfrac{-\varepsilon y}{(1 + m y)} \\[2mm] \alpha_1 z & \dfrac{\varepsilon_1 z}{(1 + m y)^2} & \alpha_1 x + \dfrac{\varepsilon_1 y}{(1 + m y)} - \mu_2 \end{bmatrix}. \tag{2}$$

Theorem 5.1. E_0 *is Unstable.*

Proof. At the point $E_0(0, 0, 0)$ in (2), we have

$$J(E_0) = \begin{bmatrix} r - \beta & 0 & 0 \\ \beta & -\mu_1 - h & 0 \\ 0 & 0 & -\mu_2 \end{bmatrix}.$$

The corresponding eigevalues are $r - \beta, -\mu_1 - h, -\mu_2$. The equilibrium point E_0 is locally stable if $r < \beta$. E_0 is unstable if $r > \beta$. $\qquad \square$

Theorem 5.2. E_1 *is Unstable.*

Proof. At position $E_1(x^*, 0, 0)$ in (2), we have

$$J(E_1) = \begin{bmatrix} -r + \beta & 0 & -\alpha k \left(1 - \dfrac{\beta}{r}\right) \\[2mm] \beta & -\mu_1 - h & 0 \\[2mm] 0 & 0 & \alpha_1 k \left(1 - \dfrac{\beta}{r}\right) - \mu_2 \end{bmatrix}.$$

The associated eigenvalues are $-\mu_1 - h, -r + \beta, \alpha_1 k \left(1 - \dfrac{\beta}{r}\right) - \mu_2$.

E_1, the equilibrium point, is not stable. The equilibrium point E_1 is stable only if $r > \beta$, $\alpha_1 k \left(1 - \dfrac{\beta}{r}\right) < \mu_2$. $\qquad \square$

Theorem 5.3. E_2 *is asymptotically stable if $A_1 A_2 - A_3 > 0$, $A_3 > 0$, $A_1 > 0$.*

Proof. At the point $E_2\left(0, y^*, z^*\right)$ in (2), we have

$$
J\left(E_2\right) = \begin{bmatrix} m_{11} & 0 & 0 \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & 0 \end{bmatrix},
$$

where

$$
m_{11} = r - \beta - a\left(\frac{-\varepsilon_1\left(\mu_1 + h\right)}{\varepsilon\left(\varepsilon_1 - m\mu_2\right)}\right), \qquad m_{21} = \beta,
$$

$$
m_{22} = \frac{-m\mu_2\left(\mu_1 + h\right)}{\varepsilon_1}, \qquad m_{23} = \frac{-\varepsilon\mu_2}{\varepsilon_1},
$$

$$
m_{31} = a_1\left(\frac{-\varepsilon_1\left(\mu_1 + h\right)}{\varepsilon\left(\varepsilon_1 - m\mu_2\right)}\right), \qquad m_{32} = \frac{\left(-\mu_1 - h\right)\left(\varepsilon_1 - m\mu_2\right)}{\varepsilon}.
$$

The associated characteristic equation is expressed as $\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0$ where,

$$
\begin{aligned}
A_1 &= -\left(m_{11} + m_{22}\right) \\
&= -\left(r - \beta - a\left(\frac{-\varepsilon_1\left(\mu_1 + h\right)}{\varepsilon\left(\varepsilon_1 - m\mu_2\right)}\right) - \frac{m\mu_2\left(\mu_1 + h\right)}{\varepsilon_1}\right),
\end{aligned}
$$

$$
\begin{aligned}
A_2 &= m_{11}m_{22} - m_{23}m_{32} \\
&= \left(\left(r - \beta + \frac{a\varepsilon_1\left(\mu_1 + h\right)}{\varepsilon\left(\varepsilon_1 - m\mu_2\right)}\right)\left(\frac{-m\mu_2\left(\mu_1 + h\right)}{\varepsilon_1}\right)\right) - \left(\frac{\left(-\mu_1 - h\right)\left(\varepsilon_1 - m\mu_2\right)\left(-\varepsilon\mu_2\right)}{\varepsilon\varepsilon_1}\right),
\end{aligned}
$$

$$
\begin{aligned}
A_3 &= m_{11}m_{23}m_{32} \\
&= \left(\left(r - \beta - a\left(\frac{-\varepsilon_1\left(\mu_1 + h\right)}{\varepsilon\left(\varepsilon_1 - m\mu_2\right)}\right)\right)\left(\frac{\left(-\varepsilon\mu_2\right)\left(-\mu_1 - h\right)\left(\varepsilon_1 - m\mu_2\right)}{\varepsilon\varepsilon_1}\right)\right).
\end{aligned}
$$

If $A_1A_2 - A_3 > 0$, $A_3 > 0$, $A_1 > 0$, then all the eigenvalues of $J\left(E_2\right)$ have real components with negative values according to Routh Hurwitz theorem. As a result, the equilibrium point E_2 exhibits asymptotic stability. □

Theorem 5.4. *E_3 is asymptotically stable if $B_1 > 0$, $B_3 > 0$, $B_1B_2 - B_3 > 0$.*

Proof. At the point $E_3\left(x^*, y^*, 0\right)$ in (2) we have

$$
J\left(E_3\right) = \begin{bmatrix} n_{11} & 0 & n_{13} \\ n_{21} & n_{22} & n_{23} \\ 0 & 0 & n_{33} \end{bmatrix},
$$

where

$$
n_{11} = r\left(1 - \frac{2x^*}{k}\right) - \beta, \quad n_{13} = -\alpha x^*, \quad n_{21} = \beta, \quad n_{22} = -\mu_1 - h,
$$

$$
n_{23} = \frac{-\varepsilon y^*}{\left(1 + my^*\right)}, \qquad n_{33} = \alpha_1 x^* + \frac{\varepsilon_1 y^*}{\left(1 + my^*\right)} - \mu_2
$$

The resulting characteristic equation is expressed as $\lambda^3 + B_1\lambda^2 + B_2\lambda + B_3 = 0$ where,

$$
n_{11} = r\left(1 - \frac{2x^*}{k}\right) - \beta, \quad n_{13} = -\alpha x^*, \quad n_{21} = \beta, \quad n_{22} = -\mu_1 - h,
$$

$$
n_{23} = \frac{-\varepsilon y^*}{\left(1 + my^*\right)}, \qquad n_{33} = \alpha_1 x^* + \frac{\varepsilon_1 y^*}{\left(1 + my^*\right)} - \mu_2.
$$

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The resulting characteristic equation is expressed as $\lambda^3 + B_1\lambda^2 + B_2\lambda + B_3 = 0$ where,

$$
\begin{aligned}
B_1 &= -(n_{11} + n_{22} + n_{33}) \\
&= -\left(r\left(1 - \frac{2x^*}{k}\right) - \beta - \mu_1 - h + \alpha_1 x^* + \frac{\varepsilon_1 y^*}{(1 + my^*)} - \mu_2 \right),
\end{aligned}
$$

$$
\begin{aligned}
B_2 &= n_{11}n_{22} + n_{11}n_{33} + n_{22}n_{33} \\
&= \left(r\left(1 - \frac{2x^*}{k}\right) - \beta \right)(-\mu_1 - h) + \left(\alpha_1 x^* + \frac{\varepsilon_1 y^*}{(1 + my^*)} - \mu_2 \right)(-\mu_1 - h) \\
&\quad + \left(r\left(1 - \frac{2x^*}{k}\right) - \beta \right)\left(\alpha_1 x^* + \frac{\varepsilon_1 y^*}{(1 + my^*)} - \mu_2 \right),
\end{aligned}
$$

$$
\begin{aligned}
B_3 &= -n_{11}n_{22}n_{33} \\
&= -\left(\left(r\left(1 - \frac{2x^*}{k}\right) - \beta \right)(-\mu_1 - h)\left(\alpha_1 x^* + \frac{\varepsilon_1 y^*}{(1 + my^*)} - \mu_2 \right) \right).
\end{aligned}
$$

If $B_1 > 0$, $B_3 > 0$, $B_1 B_2 - B_3 > 0$, then all the eigenvalues of $J(E_3)$ have real components with negative values according to Routh Hurwitz theorem. As a result, the equilibrium point E_3 is asymptotically stable. $\qquad\square$

Theorem 5.5. E_4 is asymptotically stable if $C_1 > 0$, $C_3 > 0$, $C_1 C_2 - C_3 > 0$.

Proof. At the point $E_4(x^*, y^*, z^*)$ in (2) we have

$$
J(E_4) = \begin{bmatrix} l_{11} & 0 & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{bmatrix},
$$

where

$$
l_{11} = r\left(1 - \frac{2x^*}{k}\right) - \beta - \alpha z^*, \quad l_{13} = -\alpha x^*, l_{21} = \beta, \quad l_{22} = \frac{-\varepsilon z^*}{(1 + my^*)^2} - \mu_1 - h,
$$

$$
l_{23} = \frac{-\varepsilon y^*}{(1 + my^*)}, \quad l_{31} = \alpha_1 z^*, \quad l_{32} = \frac{\varepsilon_1 z^*}{(1 + my^*)^2}, \quad l_{33} = \alpha_1 x^* + \frac{\varepsilon_1 y^*}{(1 + my^*)} - \mu_2.
$$

The corresponding characteristic equation is given as $\lambda^3 + C_1\lambda^2 + C_2\lambda + C_3 = 0$ where,

$$
\begin{aligned}
C_1 &= -(l_{11} + l_{22} + l_{33}) \\
&= -\left(r\left(1 - \frac{2x^*}{k}\right) - \beta - \alpha z^* \frac{-\varepsilon z^*}{(1 + my^*)^2} - \mu_1 - h + \alpha_1 x^* + \frac{\varepsilon_1 y^*}{(1 + my^*)} - \mu_2 \right),
\end{aligned}
$$

$$
\begin{aligned}
C_2 &= (l_{11}l_{22} + l_{11}l_{33} + l_{22}l_{33}) - (l_{31}l_{13} + l_{23}l_{32}) \\
&= \left(r\left(1 - \frac{2x^*}{k}\right) - \beta - \alpha z^* \right)\left(\frac{-\varepsilon z^*}{(1 + my^*)^2} - \mu_1 - h \right) + \alpha\alpha_1 z^* x^* + \frac{\varepsilon\varepsilon_1 z^* y^*}{(1 + my^*)^3} \\
&\quad + \left(r\left(1 - \frac{2x^*}{k}\right) - \beta - \alpha z^* \right)\left(\alpha_1 x^* + \frac{\varepsilon_1 y^*}{(1 + my^*)} - \mu_2 \right) \\
&\quad + \left(\frac{-\varepsilon z^*}{(1 + my^*)^2} - \mu_1 - h \right)\left(\alpha_1 x^* + \frac{\varepsilon_1 y^*}{(1 + my^*)} - \mu_2 \right),
\end{aligned}
$$

$$C_3 = (l_{23}l_{32}l_{11} + l_{13}l_{22}l_{31}) - (l_{11}l_{22}l_{33} + l_{13}l_{21}l_{32})$$

$$= \left(r\left(1 - \frac{2x^*}{k}\right) - \beta - \alpha z^* \right) \left(\frac{-\varepsilon\varepsilon_1 z^* y^*}{(1 + my^*)^3} \right) + \frac{\alpha\alpha_1\varepsilon x^* z^*}{(1 + my^*)^2} + \frac{\alpha\beta\varepsilon_1 x^* z^*}{(1 + my^*)^2}$$

$$+ \left(r\left(1 - \frac{2x^*}{k}\right) - \beta - \alpha z^* \right) \left(\frac{\varepsilon z^*}{(1 + my^*)^2} + \mu_1 + h \right) \left(\alpha_1 x^* + \frac{\varepsilon_1 y^*}{(1 + my^*)} - \mu_2 \right).$$

If $C_1 > 0, C_3 > 0, C_1 C_2 - C_3 > 0$. Consequently, all the eigenvalues of $J(E_4)$ have real components with negative values according to Routh Hurwitz theorem. As a result, the equilibrium point E_4 is asymptotically stable. □

6   Global Stability Analysis

The Lyapunov approach was used to create the Lyapunov function and analyse the global stability of the equilibrium points in (1) that satisfy the local stability. By building an appropriate Lyapunov function, we have ascertained the global stability of the system's internal equilibrium point $E_4 (x^*, y^*, z^*)$. Now, defining the Lyapunov function as follows,

$$V(x, y, z) = V_1(x, y, z) + V_2(x, y, z) + V_3(x, y, z),$$

where $V_1 = x - x^* - x^* \ln \dfrac{x}{x^*}$, $V_2 = y - y^* - y^* \ln \dfrac{y}{y^*}$, $V_3 = z - z^* - z^* \ln \dfrac{z}{z^*}$, which implies V is a continuous function on an integer R_+^3. Let us take the derivative of the V along with the time given as,

$$\frac{dV}{dt} = \frac{x - x^*}{x}\frac{dx}{dt} + \frac{y - y^*}{y}\frac{dy}{dt} + \frac{z - z^*}{z}\frac{dz}{dt},$$

$$\frac{1}{x}\frac{dx}{dt} = r\left(1 - \frac{x}{k}\right) - \beta - \alpha z,$$

$$\frac{1}{y}\frac{dy}{dt} = \frac{\beta x}{y} - \frac{\varepsilon z}{1 + my} - \mu_1 - h,$$

$$\frac{1}{z}\frac{dz}{dt} = \alpha_1 x + \frac{\varepsilon_1 y}{1 + my} - \mu_2,$$

$$V(x, y, z) = (x - x^*)\left[r\left(1 - \frac{x}{k}\right) - \beta - \alpha z \right] + (y - y^*)\left[\frac{\beta x}{y} - \frac{\varepsilon z}{1 + my} - \mu_1 - h \right]$$

$$+ (z - z^*)\left[\alpha_1 x + \frac{\varepsilon_1 y}{1 + my} - \mu_2 \right],$$

$$V(x, y, z) = (x - x^*)\left[\frac{-r}{k}(x - x^*) - \alpha(z - z^*) \right] + (y - y^*)\left[\frac{\beta(x - x^*)}{(y - y^*)} - \frac{\varepsilon(z - z^*)}{1 + m(y - y^*)} \right]$$

$$+ (z - z^*)\left[\alpha_1(x - x^*) + \frac{\varepsilon_1(y - y^*)}{1 + m(y - y^*)} \right]$$

$$= \frac{-r}{k}(x - x^*)^2 + \beta(x - x^*),$$

$$\frac{dV}{dt} \leq -(x - x^*)^2.$$

It is evident from the equation above that $\dfrac{dV}{dt} \leq 0$ is a negative definite function. This demonstrates the globally asymptotically stable nature of the inner equilibrium point $E_4(x^*, y^*, z^*)$.

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7 Numerical Solution

Analytical findings are meaningless without computational. Let us now consider the computational simulation for system (1). Firstly the simulation has been done between predation parameters and carrying capacity. For different values of ε the behavior of the given system is picturised in Figures 1, 2, 3. The parameter ε plays a vital role in the functional response for system (1). We observed the impact on the dynamics of system (1) by varying the value of the capture rate parameter ε. The system's dynamic behavior is depicted in Figure 1 when $\varepsilon > k$, indicating stability. The oscillatory behavior of the system is shown in Figure 2 when $\varepsilon < k$ which is unstable. Figure 3 shows the instability of the system by considering the value of $\varepsilon = 0.47$. Thus, the population of system (1) is stable in Figure 1 while unstable in Figures 2 and 3.

The variation of the population along with time and their phase portrait is given in the Figures 5, 6, 7 and 8. Figure 4 depicts the extinction of the population at the trivial equilibrium point. It is evident from this Figure 4 that all three populations are on the verge of going extinct in terms of time. It indicates the total annihilation of life in an ecosystem if a trivial equilibrium point ever becomes stable. As far as ecology and mathematics are concerned, the trivial equilibrium point E_0 is always present but never stable. Figure 5 shows system (1) is unstable by using the set values of parameters with the initial conditions $\beta = 1.15$, $\alpha = 1.3$, $k = 1.5$, $\alpha_1 = 1.5$, $\varepsilon = 1.8$, $m = 0.1$, $h = 0.5$, $\mu_1 = 0.05$, $\mu_2 = 0.7$, $\varepsilon_1 = 0.8$, $r = 1.4$ with $[1.1 \, 0.6 \, 0.6]$ in which the equilibrium point E_1 exists with the condition $r > \beta$ and the phase portrait is derived using the identical set of data as in figure 6. The system exhibits stability in Figure 7 at the specified beginning conditions and the values of $k = 1.5$, $r = 1.4$, $h = 0.5$, $m = 0.1$, $\alpha = 1.3$, $\beta = 1.15$, $\alpha_1 = 1.5$, $\varepsilon = 1.8$, $\mu_1 = 0.05$, $\mu_2 = 0.7$, $\varepsilon_1 = 1.5$ with $[1.1 \, 0.6 \, 0.6]$ and Figure 8 illustrates the phase picture of system (1) for identical set of values.
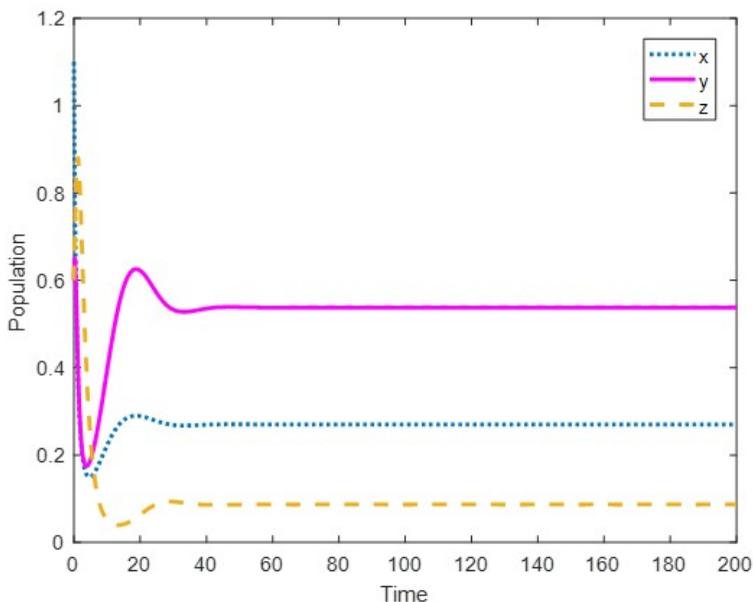

Figure 1: The behavior of system (1) with time t which is stable. Taking into account the values of $k = 1.4$, $r = 1.5$, $\beta = 1.15$, $\alpha = 0.7$, $\alpha_1 = 0.8$, $\varepsilon = 1.47$, $m = 0.1$, $h = 0.5$, $\mu_1 = 0.05$, $\mu_2 = 0.7$, $\varepsilon_1 = 0.8$.

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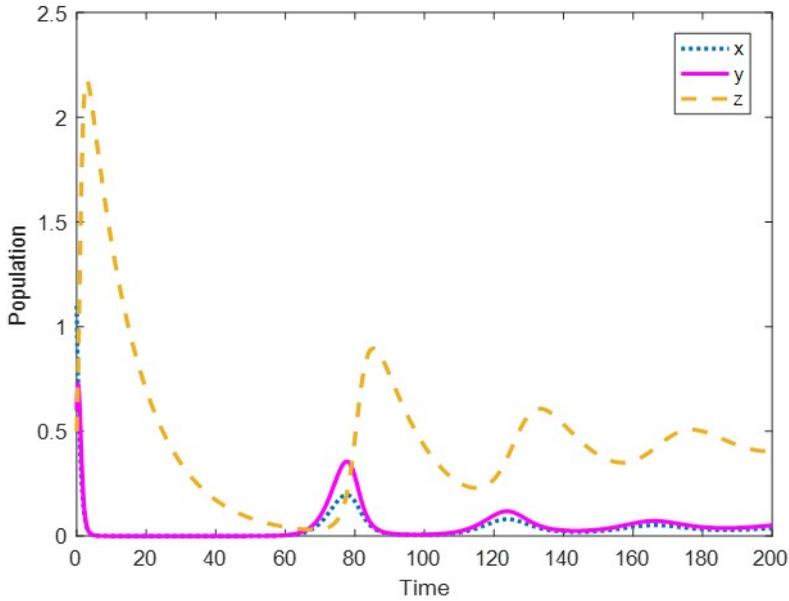*Malaysian J. Math. Sci. 19(4): 1507–1524(2025) 1507 - 1524*

Figure 2: System (1) exihibits unstable oscillatory behavior with time t . By using the values of $r = 1.5, k = 1.4, h = 0.5, m = 0.1$, $\beta = 1.15, \alpha = 0.7, \alpha_1 = 0.8, \varepsilon = 0.9, \mu_1 = 0.05, \mu_2 = 0.7, \varepsilon_1 = 0.8$.
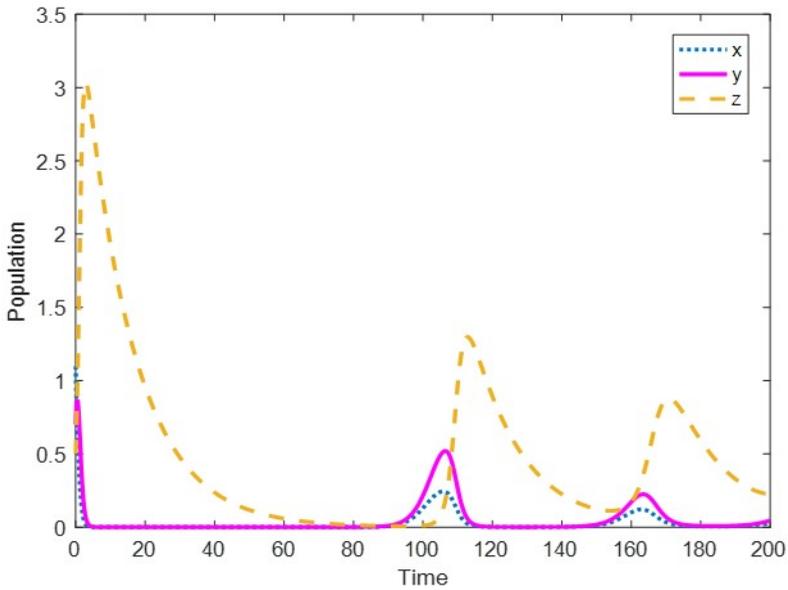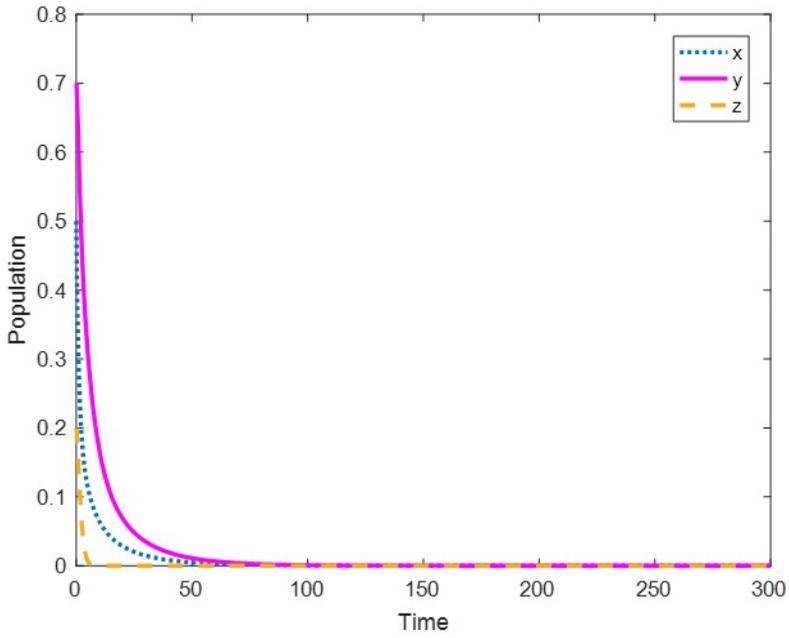

Figure 3: System (1) exihibits unstable behavior with time t . Taking into account the values of $h = 0.5, m = 0.1, r = 1.5, k = 1.4$, $\beta = 1.15, \alpha = 0.7, \alpha_1 = 0.8, \varepsilon = 0.47, \mu_1 = 0.05, \mu_2 = 0.7, \varepsilon_1 = 0.8$.

Figure 4: Extinction of the three population.
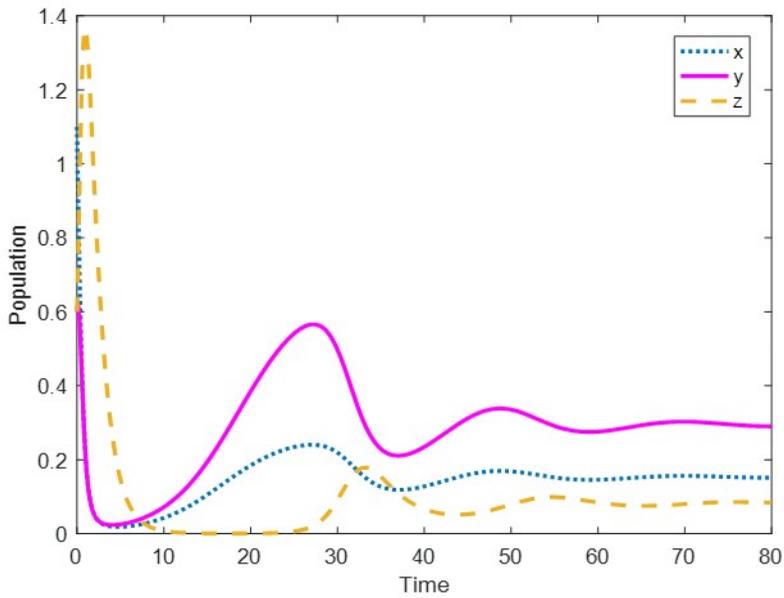
Figure 5: The variations in the densities of $z(t)$, $y(t)$ and $x(t)$ for system (1) which is unstable.

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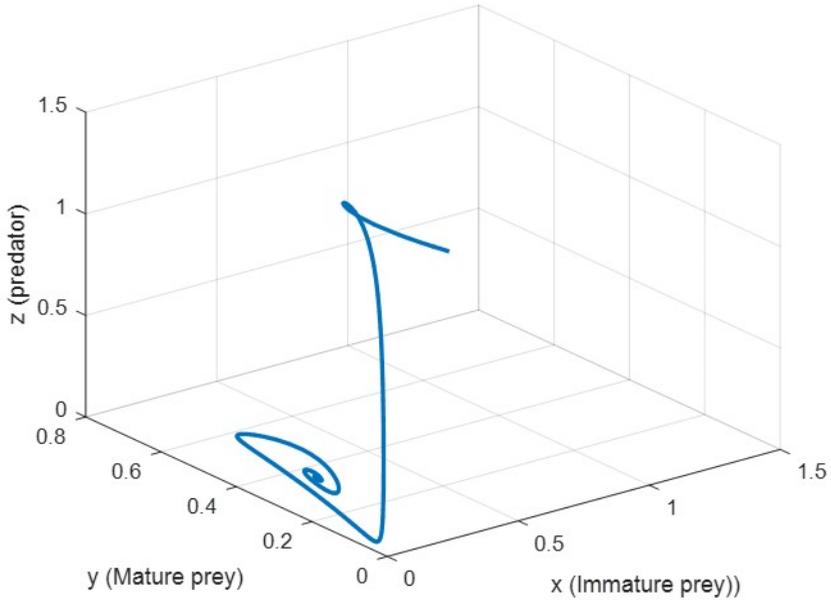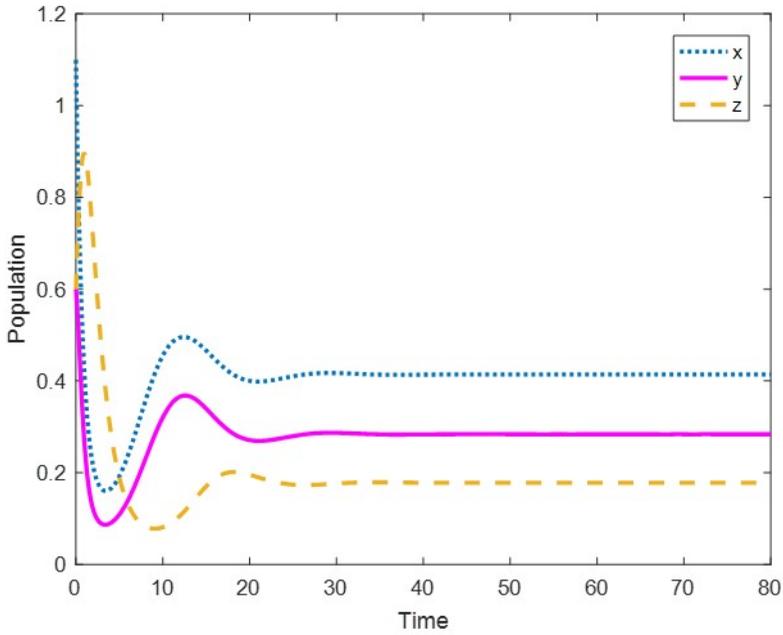

Figure 6: Phase diagram of system (1).

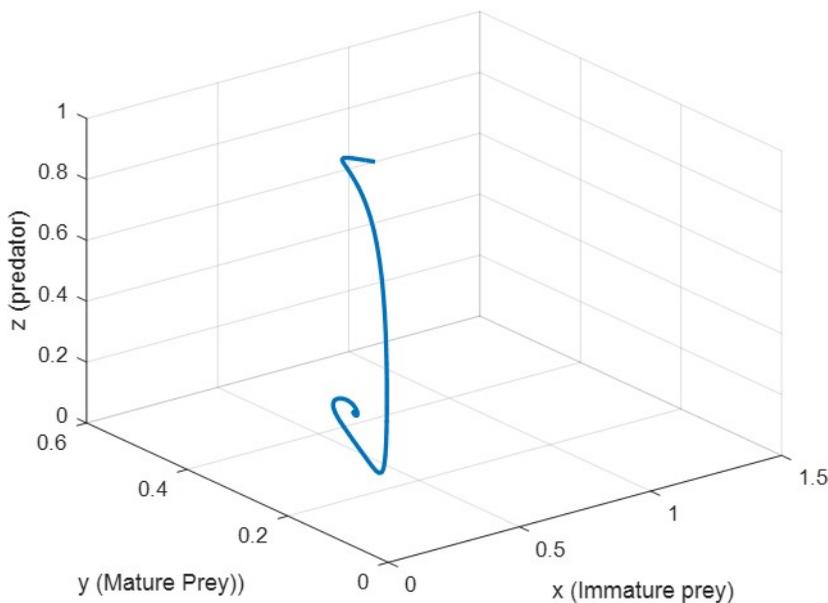Figure 7: The variations in the densities of $x(t)$, $y(t)$ and $z(t)$ is stable for system (1).

Figure 8: The phase portrait of system (1).

This section illustrates how the simulations support or uncover system behaviors, including oscillations and equilibrium points, and how these patterns relate to actual ecological observations. Here, the stability of the system is impacted by changes in the parameters, particularly the maturation and predation rates. To demonstrate the system's stability and convergence, a numerical simulation displays time series and phase diagrams. In terms of biology, the phases that deal with survival have an impact on sustainability over the long run. Harvesting is determined by the abundance of predators or the time it takes for prey to mature.

8   Conclusion

The stage-structured model for prey is taken into consideration in this paper, and prey density is split into two categories: immature and mature prey. Holling type functional response is used to facilitate interactions between these species. The process of harvesting changes the population dynamics. The mature prey has been harvested further. Positivity and boundedness are examined. All possible equilibrium points are determined. The stability analysis of the system is carried out using the Routh-Hurwitz criterion theorem. The behavior of the system with different values of parameters is picturized using MATLAB software. Thus, incorporation of different functional response is to illustrate variations in the way predator consumption rates vary with prey density, particularly as a result of handling time constraints and predator behavior. The system is able to maintain a biologically sustainable balance through the use of selective harvesting techniques that target immature prey or apply mild harvesting pressure. The biological model's interpretation of the effects of harvesting is more interesting economically, and it can be used in fisheries and forest management. For example in fishery management choosing whether to harvest immature fish, which mature later, or mature fish, which reproduce, while protecting predatory species like sharks and seals.

Conflicts of Interest There is no conflict of interest with this paper, according to the authors.

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